

Hopf hypersurfaces in complex Grassmannians of rank two

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Abstract

In this paper, we study real hypersurfaces in complex Grassmannians of rank two. First, the nonexistence of mixed foliate real hypersurfaces is proven. With this result, we show that for Hopf hypersurfaces in complex Grassmannians of rank two, the Reeb principal curvature is constant along integral curves of the Reeb vector field. As a result the classification of contact real hypersurfaces is obtained. We also introduce the notion of q -umbilical real hypersurfaces in complex Grassmannians of rank two and obtain a classification of such real hypersurfaces.

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1 Introduction

The complex Grassmannians of rank two (both the compact type: $SU_{m+2}/S(U_2U_m)$ and the noncompact type: $SU_{2,m}/S(U_2U_m)$) of complex dimension $2m$ are Riemannian symmetric spaces equipped with a Kähler structure J and a quaternionic Kähler structure \mathfrak{J} . Another interesting characteristic is the presence of the real structure JJ_a , $a \in \{1, 2, 3\}$, on its tangent spaces, arisen from the interaction between J and \mathfrak{J} , where $\{J_1, J_2, J_3\}$ is a canonical local basis for \mathfrak{J} .

These three geometric structures significantly impose restrictions on the geometry of a real hypersurface M in complex Grassmannians of rank two. As an immediate

consequence of the Codazzi equation of such submanifolds, the totally umbilicity are too strong to be satisfied by real hypersurfaces in complex Grassmannians of rank two.

Apart from the submanifold structure, three additional structures are then induced on M by these geometric structures of the ambient spaces: an almost contact structure (ϕ, ξ, η) on M from the Kähler structure J ; an almost contact 3-structure (ϕ_a, ξ_a, η_a) , $a \in \{1, 2, 3\}$ from the quaternionic Kähler structure \mathfrak{J} ; and local endomorphisms $\theta_a := \phi_a \phi - \xi_a \otimes \eta$ on TM , $a \in \{1, 2, 3\}$, from the interaction between J and \mathfrak{J} .

The formulations of the induced almost contact structure and almost contact 3-structure of real hypersurfaces M were well established and have been widely used in studying the geometry of real hypersurfaces in the literature. In contrast, less is known about the characteristics of the local endomorphism θ_a . In this paper, we establish a complete algebraic formulation for θ_a . With this notion, we introduce a concept, which is so-called q -umbilicity. To some extent, q -umbilical real hypersurfaces are those in complex Grassmannians of rank two with the richest geometric characteristics due to the nonexistence of totally umbilical real hypersurfaces.

A real hypersurface M in complex Grassmannians of rank two is said to be q -umbilical if the shape operator A of M satisfies

$$A = f_1 \mathbb{I} + f_2 \theta + f_3 \sum_{a=1}^3 \xi_a \otimes \eta_a$$

where f_1, f_2, f_3 are functions on M and $\theta := \sum_{a=1}^3 \eta_a(\xi) \theta_a$.

The concept of q -umbilicity was formulated in such a way after having taken into account the restrictions on M imposed by the three geometric structures of the ambient spaces. The absence of an almost contact structure on M under this condition is justified in the last section.

This paper is organized as follows: After a quick revision on the geometric structures on complex Grassmannians of rank two in Sect. 2 and some well-known structural equations on its real hypersurface M in the first half of Sect. 3, we establish some fundamental equations regarding the local endomorphism θ_a in the second half of Sect. 3. We also introduce an endomorphism θ on TM and obtain some of its properties in Sect. 3. In Sect. 4, we focus on Hopf hypersurfaces in complex Grassmannians of rank two. Some characteristics of the principal curvatures and the corresponding principal curvature spaces for Hopf hypersurfaces are studied. The nonexistence of mixed foliate real hypersurfaces in complex Grassmannians of rank two is obtained in Sect. 5. As an application of this result, we show that for Hopf hypersurfaces in complex Grassmannians of rank two, the Reeb principal curvature is constant along integral curves of the Reeb vector field. As a result, we can complete the classification of contact real hypersurfaces in $SU_{2,m}/S(U_2 U_m)$ initiated in [2]. In the last section, we classify q -umbilical real hypersurfaces in complex Grassmannians of rank two.

2 The complex Grassmannians of rank two

We recall some geometric structures on complex Grassmannian of rank two in this section (see [1]–[4] for details).

The complex Grassmannian $SU_{m+2}/S(U_2U_m)$ of all complex two-dimensional linear subspaces in \mathbb{C}^{m+2} is a connected, simply connected irreducible Riemannian symmetric space of compact type and with rank two. Let $G = SU_{m+2}$ and $K = S(U_2U_m)$. Denote by \mathfrak{g} and \mathfrak{k} the corresponding Lie algebra. Let $\mathfrak{m} = \mathfrak{k}^\perp$ with respect to the Killing form B of \mathfrak{g} . Then \mathfrak{m} is $\text{Ad}(K)$ -invariant and we obtain a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. The negative of B defines a positive definite inner product on \mathfrak{m} . Denote by g the corresponding G -invariant Riemannian metric on $SU_{m+2}/S(U_2U_m)$, we rescale g such that the maximal sectional curvature of $SU_{m+2}/S(U_2U_m)$ is $8c$, where $c > 0$ is a constant.

The Lie algebra \mathfrak{k} decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{R}$, where \mathfrak{R} is the center of \mathfrak{k} . Viewing \mathfrak{k} as the holonomy algebra $SU_{m+2}/S(U_2U_m)$, the center \mathfrak{R} induces a Kähler structure J , and \mathfrak{su}_2 induces a quaternionic Kähler structure \mathfrak{J} on $SU_{m+2}/S(U_2U_m)$.

The complex Grassmannian $SU_{2,m}/S(U_2U_m)$ of all positive definite complex two-dimensional linear subspaces in \mathbb{C}_2^{m+2} is a connected, simply connected irreducible Riemannian symmetric space of noncompact type and with rank two. Let $G = SU_{2,m}$ and $K = S(U_2U_m)$. Denote by \mathfrak{g} and \mathfrak{k} the corresponding Lie algebra. Consider the Cartan involution σ on \mathfrak{g} given by $\sigma(A) = SAS^{-1}$, where $S = \begin{bmatrix} -I_2 & 0 \\ 0 & I_m \end{bmatrix}$. Then $B_\sigma(X, Y) := -B(X, \sigma Y)$ is a positive definite $\text{Ad}(K)$ -invariant inner product on \mathfrak{g} , where B is the Killing form of G . Let $(\mathfrak{k}, \mathfrak{m})$ be a Cartan pair of \mathfrak{g} associated to the Cartan involution σ . Then the restriction of B_σ to \mathfrak{m} induces a Riemannian metric g on $SU_{2,m}/S(U_2U_m)$, which is unique up to a scaling. We take a scaling factor $c < 0$ such that the minimal sectional curvature of $SU_{2,m}/S(U_2U_m)$ is $8c$.

The Lie algebra \mathfrak{k} can be decomposed orthogonally as $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where \mathfrak{u}_1 is the center of \mathfrak{k} . The adjoint action of \mathfrak{su}_2 on \mathfrak{m} induces a quaternionic Kähler structure \mathfrak{J} , and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2}I_2 & 0 \\ 0 & \frac{-2i}{m+2}I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces a Kähler structure J on $SU_{2,m}/S(U_2U_m)$ respectively.

In this paper, we use a unified notation. Denote by $\hat{M}^m(c)$ the compact complex Grassmannian $SU_{m+2}/S(U_2U_m)$ of rank two (resp. noncompact complex Grassmannian $SU_{2,m}/S(U_2U_m)$ of rank two) for $c > 0$ (resp. $c < 0$), where c is a scaling factor for the Riemannian metric g .

For each $x \in \hat{M}^m(c)$, let $\{J_1, J_2, J_3\}$ be a canonical local basis of \mathfrak{J} on a neighborhood \mathcal{U} of $x \in \hat{M}(c)$, that is, each J_a is a local almost Hermitian structure such that

$$J_a J_{a+1} = J_{a+2} = -J_{a+1} J_a, \quad a \in \{1, 2, 3\}. \quad (2.1)$$

Here, the index a is taken modulo three. Denote by $\hat{\nabla}$ the Levi-Civita connection of $\hat{M}^m(c)$. There exists local 1-forms q_1, q_2 and q_3 such that

$$\hat{\nabla}_X J_a = q_{a+2}(X) J_{a+1} - q_{a+1}(X) J_{a+2}$$

for any $X \in T_x \hat{M}^m(c)$, that is, \mathfrak{J} is parallel with respect to $\hat{\nabla}$. The Kähler structure J and quaternionic Kähler structure \mathfrak{J} are related by

$$JJ_a = J_a J; \quad \text{Trace}(JJ_a) = 0, \quad a \in \{1, 2, 3\}. \quad (2.2)$$

The Riemannian curvature tensor \hat{R} of $\hat{M}(c)$ is locally given by

$$\begin{aligned} \hat{R}(X, Y)Z = & c\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ & - g(JX, Z)JY - 2g(JX, Y)JZ\} \\ & + c \sum_{a=1}^3 \{g(J_a Y, Z)J_a X - g(J_a X, Z)J_a Y - 2g(J_a X, Y)J_a Z \\ & + g(JJ_a Y, Z)JJ_a X - g(JJ_a X, Z)JJ_a Y\}. \end{aligned} \quad (2.3)$$

for all X, Y and $Z \in T_x \hat{M}^m(c)$.

For a nonzero vector $X \in T_x \hat{M}^m(c)$, we denote by $\mathfrak{J}X = \{J'X | J' \in \mathfrak{J}_x\}$. Recall that a maximal flat in a $\hat{M}^m(c)$ is a connected complete, totally geodesic flat submanifold of maximal dimension. Let $X \in T\hat{M}^m(c)$ be a non-zero vector. Then X is said to be *singular* if it is contained in more than one maximal flat in $\hat{M}^m(c)$. It is well-known that X is singular if and only if either $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

3 Real hypersurfaces in $\hat{M}^m(c)$

In this section, we prepare and derive some fundamental identities for real hypersurfaces in $\hat{M}^m(c)$. Some of these identities have been proven in [2, 3, 4, 10, 11]. Some well-known results are also stated.

Let M be a connected, oriented real hypersurface isometrically immersed in $\hat{M}^m(c)$, $m \geq 3$, and N a unit normal vector field on M . Denote by the same g the Riemannian metric on M . A canonical local basis $\{J_1, J_2, J_3\}$ of \mathfrak{J} on $\hat{M}^m(c)$ induces an almost contact metric 3-structure $(\phi_a, \xi_a, \eta_a, g)$ on M by

$$J_a X = \phi_a X + \eta_a(X)N, \quad J_a N = -\xi_a, \quad \eta_a(X) = g(X, \xi_a),$$

for any $X \in TM$. It follows that

$$\left. \begin{aligned} \phi_a \phi_{a+1} - \xi_a \otimes \eta_{a+1} &= \phi_{a+2} = -\phi_{a+1} \phi_a + \xi_{a+1} \otimes \eta_a \\ \phi_a \xi_{a+1} &= \xi_{a+2} = -\phi_{a+1} \xi_a \end{aligned} \right\} \quad (3.1)$$

for $a \in \{1, 2, 3\}$. The indices in the preceding equations are taken modulo three.

Let (ϕ, ξ, η, g) be the almost contact metric structure on M induced by J , that is,

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = g(X, \xi).$$

The two structures (ϕ, ξ, η, g) and $(\phi_a, \xi_a, \eta_a, g)$ are related as follows

$$\phi_a \phi - \xi_a \otimes \eta = \phi \phi_a - \xi \otimes \eta_a; \quad \phi \xi_a = \phi_a \xi. \quad (3.2)$$

Next, we denote by ∇ the Levi-Civita connection and A the shape operator on M . Then

$$\left. \begin{aligned} (\nabla_X \phi)Y &= \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX \\ (\nabla_X \phi_a)Y &= \eta_a(Y)AX - g(AX, Y)\xi_a + q_{a+2}(X)\phi_{a+1}Y - q_{a+1}(X)\phi_{a+2}Y \\ \nabla_X \xi_a &= \phi_a AX + q_{a+2}(X)\xi_{a+1} - q_{a+1}(X)\xi_{a+2} \\ X\eta(\xi_a) &= 2\eta_a(\phi AX) + \eta_{a+1}(\xi)q_{a+2}(X) - \eta_{a+2}(\xi)q_{a+1}(X). \end{aligned} \right\} \quad (3.3)$$

Let $\mathfrak{D}^\perp = \mathfrak{J}N$, and \mathfrak{D} its orthogonal complement in TM . If $\xi \in \mathfrak{D}$, then $\eta(\xi_a) = 0$ for $a \in \{1, 2, 3\}$ and so by the preceding equation, we obtain

Lemma 3.1. *If $\xi \in \mathfrak{D}$, then $A\phi\xi_a = 0$ for $a \in \{1, 2, 3\}$.*

We define a local symmetric $(1, 1)$ -tensor field θ_a on M by

$$\theta_a := \phi_a \phi - \xi_a \otimes \eta.$$

Then we have the following identities

Lemma 3.2. *(a) θ_a is symmetric,*

- (b) $\text{Trace}(\theta_a) = \eta(\xi_a)$,
- (c) $\theta_a \xi = -\xi_a$; $\theta_a \xi_a = -\xi$; $\theta_a \phi \xi_a = \eta(\xi_a) \phi \xi_a$,
- (d) $\theta_a \xi_{a+1} = \phi \xi_{a+2} = -\theta_{a+1} \xi_a$,
- (e) $\theta_a^2 - \phi \xi_a \otimes \eta_a \phi = \mathbb{I}$,
- (f) $-\theta_a \phi \xi_{a+1} + \eta(\xi_{a+1}) \phi \xi_a = \xi_{a+2} = \theta_{a+1} \phi \xi_a - \eta(\xi_a) \phi \xi_{a+1}$,
- (g) $-\theta_a \theta_{a+1} + \phi \xi_a \otimes \eta_{a+1} \phi = \phi_{a+2} = \theta_{a+1} \theta_a - \phi \xi_{a+1} \otimes \eta_a \phi$,
- (h) $\theta_a \phi - \phi \xi_a \otimes \eta = -\phi_a = \phi \theta_a - \xi \otimes \eta_a \phi$,
- (i) $\theta_a \phi_a - \phi \xi_a \otimes \eta_a = -\phi = \phi_a \theta_a - \xi_a \otimes \eta_a \phi$,
- (j) $\theta_a \phi_{a+1} - \phi \xi_a \otimes \eta_{a+1} = \theta_{a+2} = -\phi_{a+1} \theta_a - \xi_{a+1} \otimes \eta_a \phi$.

Proof. (a)–(f) The proof is exactly the same as that given in [10].

(g) By using (3.1)–(3.2), we have

$$\begin{aligned} \theta_a \theta_{a+1} - \phi \xi_a \otimes \eta_{a+1} \phi &= (\phi_a \phi - \xi_a \otimes \eta)(\phi \phi_{a+1} - \xi \otimes \eta_{a+1}) - \phi \xi_a \otimes \eta_{a+1} \phi \\ &= \phi_a \phi^2 \phi_{a+1} + \xi_a \otimes \eta_{a+1} - \phi \xi_a \otimes \eta_{a+1} \phi \\ &= \phi_a (-\mathbb{I} + \xi \otimes \eta) \phi_{a+1} + \xi_a \otimes \eta_{a+1} - \phi \xi_a \otimes \eta_{a+1} \phi \\ &= -\phi_a \phi_{a+1} + \xi_a \otimes \eta_{a+1} \\ &= -\phi_{a+2}. \end{aligned}$$

The second equality can be obtained as follows

$$\phi_{a+2} = (-\phi_{a+2})^* = (\theta_a \theta_{a+1} - \phi \xi_a \otimes \eta_{a+1} \phi)^* = \theta_{a+1} \theta_a - \phi \xi_{a+1} \otimes \eta_a \phi$$

where we denote by T^* the adjoint of an endomorphism T with respect to g .

(h)–(j) The first equalities can be obtained as follows

$$\begin{aligned}
\theta_a \phi &= (\phi_a \phi - \xi_a \otimes \eta) \phi = \phi_a(-\mathbb{I} + \xi \otimes \eta) = -\phi_a + \phi \xi_a \otimes \eta \\
\theta_a \phi_a &= (\phi \phi_a - \xi \otimes \eta_a) \phi_a = \phi(-\mathbb{I} + \xi_a \otimes \eta_a) = -\phi + \phi \xi_a \otimes \eta_a \\
\theta_a \phi_{a+1} &= (\phi \phi_a - \xi \otimes \eta_a) \phi_{a+1} = \phi(\phi_{a+2} + \xi_a \otimes \eta_{a+1}) - \xi \otimes \eta_{a+2} \\
&= (\phi \phi_{a+2} - \xi \otimes \eta_{a+2}) + \phi \xi_a \otimes \eta_{a+1} = \theta_{a+2} + \phi \xi_a \otimes \eta_{a+1}.
\end{aligned}$$

In a similar manner as in (g), we can obtain the second equalities for these parts. \square

Note that

$$\begin{aligned}
(\nabla_X \theta_a) Y &= (\nabla_X \phi) \phi_a Y + \phi(\nabla_X \phi_a) Y - g(\nabla_X \xi_a, Y) \xi - \eta_a(Y) \nabla_X \xi \\
\nabla_X \phi \xi_a &= (\nabla_X \phi) \xi_a + \phi \nabla_X \xi_a.
\end{aligned}$$

Then by applying (3.3), we obtain

$$\left. \begin{aligned}
(\nabla_X \theta_a) Y &= \eta_a(\phi Y) A X - g(A X, Y) \phi \xi_a + q_{a+2}(X) \theta_{a+1} Y - q_{a+1}(X) \theta_{a+2} Y \\
\nabla_X \phi \xi_a &= \theta_a A X + \eta_a(\xi) A X + q_{a+2}(X) \phi \xi_{a+1} - q_{a+1}(X) \phi \xi_{a+2}.
\end{aligned} \right\} \quad (3.4)$$

For each $x \in M$, we define a subspace \mathcal{H}^\perp of $T_x M$ by

$$\mathcal{H}^\perp := \text{span}\{\xi, \xi_1, \xi_2, \xi_3, \phi \xi_1, \phi \xi_2, \phi \xi_3\}.$$

Let \mathcal{H} be the orthogonal complement of \mathcal{H}^\perp in $T_x M$. Then $\dim \mathcal{H} = 4m - 4$ (resp. $\dim \mathcal{H} = 4m - 8$) when $\xi \in \mathfrak{D}^\perp$ (resp. $\xi \notin \mathfrak{D}^\perp$) and \mathcal{H} is invariant under ϕ, ϕ_a and θ_a . Moreover, $\theta_a|_{\mathcal{H}}$ has two eigenvalues: 1 and -1 . Denote by $\mathcal{H}_a(\varepsilon)$ the eigenspace corresponding to the eigenvalue ε of $\theta_a|_{\mathcal{H}}$. Then $\dim \mathcal{H}_a(1) = \dim \mathcal{H}_a(-1)$ is even, and

$$\left. \begin{aligned}
\phi \mathcal{H}_a(\varepsilon) &= \phi_a \mathcal{H}_a(\varepsilon) = \theta_a \mathcal{H}_a(\varepsilon) = \mathcal{H}_a(\varepsilon) \\
\phi_b \mathcal{H}_a(\varepsilon) &= \theta_b \mathcal{H}_a(\varepsilon) = \mathcal{H}_a(-\varepsilon), \quad (a \neq b).
\end{aligned} \right\} \quad (3.5)$$

The proof of (3.5) is exactly the same as presented in [10, pp. 92–93].

Observe that $\tan(JJ_a X) = \theta_a X$ and $\text{nor}(JJ_a X) = \eta_a(\phi X) N$, for $X \in TM$. Then the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned}
R(X, Y) Z &= g(A Y, Z) A X - g(A X, Z) A Y + c\{g(Y, Z) X - g(X, Z) Y \\
&\quad + g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y - 2g(\phi X, Y) \phi Z\} \\
&\quad + c \sum_{a=1}^3 \{g(\phi_a Y, Z) \phi_a X - g(\phi_a X, Z) \phi_a Y - 2g(\phi_a X, Y) \phi_a Z \\
&\quad + g(\theta_a Y, Z) \theta_a X - g(\theta_a X, Z) \theta_a Y\}
\end{aligned}$$

$$\begin{aligned}
(\nabla_X A) Y - (\nabla_Y A) X &= c\{\eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \xi\} \\
&\quad + c \sum_{a=1}^3 \{\eta_a(X) \phi_a Y - \eta_a(Y) \phi_a X - 2g(\phi_a X, Y) \xi_a \\
&\quad + \eta_a(\phi X) \theta_a Y - \eta_a(\phi Y) \theta_a X\}.
\end{aligned}$$

We define the tensor fields θ , ϕ^\perp , ξ^\perp and η^\perp on M as follows

$$\theta := \sum_{a=1}^3 \eta_a(\xi) \theta_a, \quad \phi^\perp := \sum_{a=1}^3 \eta_a(\xi) \phi_a, \quad \xi^\perp := \sum_{a=1}^3 \eta_a(\xi) \xi_a, \quad \eta^\perp := \sum_{a=1}^3 \eta_a(\xi) \eta_a.$$

Lemma 3.3. *At each $x \in M$ with $\xi^\perp \neq 0$, $\theta|_{\mathcal{H}}$ has two eigenvalues $\varepsilon \|\xi^\perp\|$, $\varepsilon \in \{1, -1\}$. Let $\mathcal{H}(\varepsilon)$ be the eigenspace of $\theta|_{\mathcal{H}}$ corresponding to $\varepsilon \|\xi^\perp\|$. Then*

$$(a) \quad \phi \mathcal{H}(\varepsilon) = \phi^\perp \mathcal{H}(\varepsilon) = \mathcal{H}(\varepsilon),$$

$$(b) \quad \dim \mathcal{H}(1) = \dim \mathcal{H}(-1) \text{ is even.}$$

Proof. We take a canonical local basis of \mathfrak{J} such that $\xi_1 = \xi^\perp / \|\xi^\perp\|$, so $\theta = \eta_1(\xi) \theta_1$, $\phi^\perp = \eta_1(\xi) \phi_1$ and $\eta^\perp = \eta_1(\xi) \eta_1$. It follows that $\mathcal{H}(\varepsilon) = \mathcal{H}_1(\varepsilon)$ and so these results follow from (3.5). \square

As in the proof of Lemma 3.3, at each $x \in M$ with $\|\xi^\perp\| \neq 0$, we can take a canonical local basis of \mathfrak{J} such that on a neighborhood $G \subset M$ of x , we have

$$\left. \begin{aligned} \xi_1 &= \frac{\xi^\perp}{\|\xi^\perp\|}, \quad 0 < \eta_1(\xi) = \|\xi^\perp\| \leq 1, \quad \eta_2(\xi) = \eta_3(\xi) = 0 \\ \mathcal{H}(\varepsilon) &= \mathcal{H}_1(\varepsilon), \quad \theta = \eta_1(\xi) \theta_1, \quad \phi^\perp = \eta_1(\xi) \phi_1, \quad \eta^\perp = \eta_1(\xi) \eta_1 \end{aligned} \right\} \quad (3.6)$$

where $\mathcal{H}(\varepsilon)$ is the eigenspace of $\theta|_{\mathcal{H}}$ corresponding to an eigenvalue $\varepsilon \|\xi^\perp\|$ of $\theta|_{\mathcal{H}}$ for $\varepsilon \in \{1, -1\}$. Furthermore if $\|\xi^\perp\| = 1$ at x , then

$$\xi_1 = \xi = \xi^\perp, \quad \xi_2 = \theta \xi_2 = \phi \xi_3, \quad \xi_3 = \theta \xi_3 = -\phi \xi_2 \quad (3.7)$$

Throughout this paper, we always consider such a local orthonormal frame $\{\xi_1, \xi_2, \xi_3\}$ on \mathfrak{D}^\perp under these situations.

Lemma 3.4. (a) $\theta \xi^\perp = -\|\xi^\perp\|^2 \xi$, $\theta \xi = -\xi^\perp$, $\theta \phi \xi^\perp = \|\xi^\perp\|^2 \phi \xi^\perp$,

$$(b) \quad \theta^2 - \phi \xi^\perp \otimes \eta^\perp \phi = \|\xi^\perp\|^2 \mathbb{I},$$

$$(c) \quad \theta \phi - \phi \xi^\perp \otimes \eta = -\phi^\perp = \phi \theta - \xi \otimes \eta^\perp \phi,$$

$$(d) \quad (\phi^\perp)^2 = -\|\xi^\perp\|^2 \mathbb{I} + \xi^\perp \otimes \eta^\perp,$$

$$(e) \quad \phi^\perp \phi - \xi^\perp \otimes \eta = \theta = \phi \phi^\perp - \xi \otimes \eta^\perp,$$

$$(f) \quad d(\|\xi^\perp\|^2) = 4\eta^\perp \phi A,$$

$$(g) \quad (\nabla_X \phi^\perp) Y = \eta^\perp(Y) A X - g(A X, Y) \xi^\perp + 2 \sum_{a=1}^3 \eta_a(\phi A X) \phi_a Y$$

$$(h) \quad \nabla_X \xi^\perp = \phi^\perp A X + 2 \sum_{a=1}^3 \eta_a(\phi A X) \xi_a$$

$$(i) \quad (\nabla_X \theta) Y = \eta^\perp(\phi Y) A X - g(A X, Y) \phi \xi^\perp + 2 \sum_{a=1}^3 \eta_a(\phi A X) \theta_a Y$$

$$(j) \quad \nabla_X \phi \xi^\perp = \theta A X + \|\xi^\perp\|^2 A X + 2 \sum_{a=1}^3 \eta_a(\phi A X) \phi \xi_a.$$

Proof. (a)–(e) At each $x \in M$ with $\xi^\perp = 0$, we have $\theta = \phi^\perp = 0$ and $\eta^\perp = 0$. Hence these identities are trivial. Suppose $\|\xi^\perp\| \neq 0$ at a point $x \in M$. Then these identities can be easily obtained from Lemma 3.2 and (3.6).

(f)–(j) We only give the proof for (i) as the remaining parts can be obtained by a similar straightforward calculation. By using (3.3)–(3.4),

$$\begin{aligned} (\nabla_X \theta)Y &= \sum_{a=1}^3 \{(X(\eta_a(\xi))\theta_a Y + \eta_a(\xi)(\nabla_X \theta_a)Y\} \\ &= \sum_{a=1}^3 \{2\eta_a(\phi AX)\theta_a Y + \eta_a(\xi)\eta_a(\phi Y)AX - g(AX, Y)\eta_a(\xi)\phi_a \xi\} \\ &= \sum_{a=1}^3 2\eta_a(\phi AX)\theta_a Y + \eta^\perp(\phi Y)AX - g(AX, Y)\phi \xi^\perp. \end{aligned}$$

□

We now prepare some results for later use. By using Lemma 3.4, we have

$$\begin{aligned} d(\eta^\perp \phi)(X, Y) &= -g(\nabla_X \phi \xi^\perp, Y) + g(\nabla_Y \phi \xi^\perp, X) \\ &= -g((\theta A - A\theta)X, Y) + 2 \sum_{a=1}^3 \{\eta_a(\phi AX)\eta_a(\phi Y) - \eta_a(\phi AY)\eta_a(\phi X)\}. \end{aligned} \quad (3.8)$$

Lemma 3.5. *Suppose $0 < \|\xi^\perp\| < 1$. If $A\phi \xi^\perp = \omega \phi \xi^\perp$, then*

- (a) $d\omega = -\|\xi^\perp\|^{-2}(1 - \|\xi^\perp\|^2)^{-1}(\phi \xi^\perp \omega) \eta^\perp \phi$,
- (b) $\omega d(\eta^\perp \phi) = 0$.

Proof. It follows from the hypothesis and Lemma 3.4 that

$$d\omega \wedge \eta^\perp \phi + \omega d(\eta^\perp \phi) = \frac{1}{4} d^2(\|\xi^\perp\|^2) = 0.$$

This, together with (3.8), gives

$$X\omega = - \frac{(\phi \xi^\perp \omega) \eta^\perp(\phi X) + (d\omega \wedge \eta^\perp \phi)(X, \phi \xi^\perp)}{\|\xi^\perp\|^2(1 - \|\xi^\perp\|^2)} = - \frac{(\phi \xi^\perp \omega) \eta^\perp(\phi X)}{\|\xi^\perp\|^2(1 - \|\xi^\perp\|^2)}.$$

This means that $d\omega = -\|\xi^\perp\|^{-2}(1 - \|\xi^\perp\|^2)^{-1}(\phi \xi^\perp \omega) \eta^\perp \phi$ and hence $\omega d(\eta^\perp \phi) = 0$. □

Lemma 3.6. *Suppose $\|\xi^\perp\| = 1$ on M . Then*

- (a) $\theta A + A = -2 \sum_{a=1}^3 \phi \xi_a \otimes \eta_a \phi A$,
- (b) $A\mathcal{H}(1) = 0$.

Proof. (a) Since $\xi = \xi^\perp$,

$$0 = \phi \nabla_X (\xi^\perp - \xi) = (\theta A + A)X + 2 \sum_{a=1}^3 \eta_a(\phi AX) \phi \xi_a, \quad X \in TM.$$

(b) Let $Y \in \mathcal{H}(1)$. Then $2g(AY, X) = g(Y, (\theta A + A)X) = 0$ for $X \in TM$, which means that $A\mathcal{H}(1) = 0$. \square

Lemma 3.7. *At each $x \in M$ with $\|\xi^\perp\| > 0$,*

$$\sum_{a=1}^3 g(\theta_a X, Y) \eta_a(Z) = \sum_{a=1}^3 g(\phi_a X, Y) \eta_a(Z) = 0$$

for any $X, Y, Z \in \mathcal{H}(\varepsilon) \oplus (\mathcal{H}^\perp \ominus \text{Span}\{\xi, \xi^\perp, \phi \xi^\perp\})$, where $\varepsilon \in \{1, -1\}$.

Proof. Under the setting in (3.6), we have

$$\mathcal{H}^\perp \ominus \text{Span}\{\xi, \xi^\perp, \phi \xi^\perp\} = \text{Span}\{\xi_2, \xi_3, \phi \xi_2, \phi \xi_3\}.$$

It suffices to show that

$$g(\theta_b X, Y) = g(\phi_b X, Y) = 0, \quad X, Y \in \mathcal{H}_1(\varepsilon) \oplus \text{Span}\{\xi_2, \xi_3, \phi \xi_2, \phi \xi_3\}, b \in \{2, 3\}.$$

First, we can easily verify that

$$\theta_b \xi_c, \phi_b \xi_c, \theta_b \phi \xi_c, \phi_b \phi \xi_c \in \text{Span}\{\xi, \xi_1, \phi \xi_1\}, \quad b, c \in \{2, 3\}.$$

This, together with (3.5), gives the first equation. \square

Lemma 3.8. *Suppose $0 < \|\xi^\perp\| < 1$ on M . If $A\mathcal{H}(\varepsilon) \subset \mathcal{H}(\varepsilon)$, where $\varepsilon \in \{1, -1\}$, then $\nabla_X Y \perp \mathcal{H}^\perp \ominus \text{Span}\{\xi, \xi^\perp, \phi \xi^\perp\}$ for all vector fields X, Y tangent to $\mathcal{H}(\varepsilon)$.*

Proof. We adopt the basis $\{\xi_1, \xi_2, \xi_3\}$ for \mathfrak{D}^\perp with properties (3.6). Let X, Y be vector fields tangent to $\mathcal{H}(\varepsilon)$. Then for $b \in \{2, 3\}$, by using (3.5) we obtain

$$g(\nabla_X \xi_b, Y) = g(\phi_b AX, Y) = 0; \quad g(\nabla_X \phi \xi_b, Y) = g(\theta_b AX, Y) = 0.$$

Since $\mathcal{H}^\perp \ominus \text{Span}\{\xi, \xi^\perp, \phi \xi^\perp\} = \text{Span}\{\xi_2, \xi_3, \phi \xi_2, \phi \xi_3\}$, this gives the desired result. \square

If $\|\xi^\perp\| = 1$, then $\xi^\perp = \xi$ and $\mathcal{H}^\perp = \mathfrak{D}^\perp$. Using a similar method as in the preceding proof, we obtain

Lemma 3.9. *Suppose $\|\xi^\perp\| = 1$ on M . If $\mathcal{V} \subset \mathcal{H}(\varepsilon)$, where $\varepsilon \in \{1, -1\}$, is a subbundle that is invariant under A , then $\nabla_X Y \perp \mathfrak{D}^\perp \ominus \mathbb{R}\xi$ for all vector fields X, Y tangent to \mathcal{V} .*

At the end of this section, we state some important results for later use.

Theorem 3.1 ([3]). *Let M be a connected real hypersurface in $SU_{m+2}/S(U_2 U_m)$, $m \geq 3$. Then both $\mathbb{R}\xi$ and \mathfrak{D}^\perp are invariant under the shape operator of M if and only if M is an open part of one of the following spaces:*

- (A) a tube around a totally geodesic $SU_{m+1}/S(U_2U_{m-1})$ in $SU_{m+2}/S(U_2U_m)$, or
- (B) a tube around a totally geodesic $\mathbb{H}P_n = Sp_{n+1}/Sp_1Sp_n$ in $SU_{m+2}/S(U_2U_m)$, where $m = 2n$ is even.

The principal curvatures of real hypersurfaces of type A and B stated in Theorem 3.1 are given as follows:

Theorem 3.2 ([3]). *Given a constant $c > 0$:*

- (i) *If M is a real hypersurface of type A in $SU_{m+2}/S(U_2U_m)$, then $\xi \in \mathfrak{D}^\perp$ at each point of M , and M has three (for $r = \pi/2\sqrt{8c}$, whereby $\alpha = \mu$) or four (otherwise) distinct constant principal curvatures*

$$\begin{aligned}\alpha &= \sqrt{8c} \cot(\sqrt{8c}r), & \beta &= \sqrt{2c} \cot(\sqrt{2c}r), \\ \lambda &= -\sqrt{2c} \tan(\sqrt{2c}r), & \mu &= 0\end{aligned}$$

with some $r \in]0, \pi/\sqrt{8c}[$. The corresponding principal curvature spaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathfrak{D}^\perp \ominus \mathbb{R}\xi, \quad T_\lambda = \mathcal{H}(-1), \quad T_\mu = \mathcal{H}(1).$$

- (ii) *If M is a real hypersurface of type B in $SU_{m+2}/S(U_2U_m)$, then $\xi \in \mathfrak{D}$ at each point of M , $m = 2n$ is even and M has five distinct constant principal curvatures*

$$\begin{aligned}\alpha &= -2\sqrt{c} \tan(2\sqrt{c}r), & \beta &= 2\sqrt{c} \cot(2\sqrt{c}r), & \gamma &= 0, \\ \lambda &= \sqrt{c} \cot(\sqrt{c}r), & \mu &= -\sqrt{c} \tan(\sqrt{c}r)\end{aligned}$$

with some $r \in]0, \pi/4\sqrt{c}[$. The corresponding principal curvature spaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathfrak{D}^\perp, \quad T_\gamma = \mathfrak{J}\xi, \quad T_\lambda, \quad T_\mu,$$

where $T_\lambda \oplus T_\mu = \mathcal{H}$, $\mathfrak{J}T_\lambda = T_\lambda$, $\mathfrak{J}T_\mu = T_\mu$, $JT_\lambda = T_\mu$.

Theorem 3.3 ([4]). *Let M be a connected real hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. Then both $\mathbb{R}\xi$ and \mathfrak{D}^\perp are invariant under the shape operator of M if and only if one of the following holds:*

- (A) *M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$, or*
- (B) *M is an open part of a tube around a totally geodesic $\mathbb{H}H_n = Sp_{1,n}/Sp_1Sp_n$ in $SU_{2,m}/S(U_2U_m)$, where $m = 2n$ is even, or*
- (C₁) *M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \in \mathfrak{J}N$, or*
- (C₂) *M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$, or*
- (D) *the normal bundle of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$. Moreover, M has at least four distinct principal curvatures, which are given by*

$$\alpha = 2\sqrt{-c}, \quad \gamma = 0, \quad \lambda = \sqrt{-c}, \quad (c < 0 \text{ is a constant})$$

with corresponding principal curvature spaces

$$T_\alpha = \mathbb{R}\xi \oplus \mathfrak{D}^\perp, \quad T_\gamma = \mathfrak{J}\xi, \quad T_\lambda \subset \mathcal{H}.$$

If μ is another (possibly nonconstant) principal curvature function, then $T_\mu \subset \mathcal{H}$, $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

The principal curvatures of real hypersurfaces of type A , B , C_1 and C_2 stated in Theorem 3.3 are given as follows:

Theorem 3.4 ([4]). *Given a constant $c < 0$:*

(i) *If M is a real hypersurface of type A in $SU_{2,m}/S(U_2U_m)$, then $\xi \in \mathfrak{D}^\perp$ at each point of M , and M has four distinct constant principal curvatures*

$$\begin{aligned} \alpha &= \sqrt{-8c} \coth(\sqrt{8cr}), & \beta &= \sqrt{-2c} \coth(\sqrt{-2cr}), \\ \lambda &= \sqrt{-2c} \tanh(\sqrt{-2cr}), & \mu &= 0 \end{aligned}$$

with some $r > 0$. The corresponding principal curvature spaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathfrak{D}^\perp \ominus \mathbb{R}\xi, \quad T_\lambda = \mathcal{H}(-1), \quad T_\mu = \mathcal{H}(1).$$

(ii) *If M is a real hypersurface of type B in $SU_{2,m}/S(U_2U_m)$, then $\xi \in \mathfrak{D}$ at each point of M , $m = 2n$ is even and M has four (for $\tanh^2 \sqrt{-cr} = 1/3$, whereby $\alpha = \lambda$) or five (otherwise) distinct constant principal curvatures*

$$\begin{aligned} \alpha &= 2\sqrt{-c} \tanh(2\sqrt{-cr}), & \beta &= 2\sqrt{-c} \coth(2\sqrt{-cr}), & \gamma &= 0, \\ \lambda &= \sqrt{-c} \coth(\sqrt{-cr}), & \mu &= \sqrt{-c} \tanh(\sqrt{-cr}) \end{aligned}$$

with some $r > 0$. The corresponding principal curvature spaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathfrak{D}^\perp, \quad T_\gamma = \mathfrak{J}\xi, \quad T_\lambda, \quad T_\mu,$$

where $T_\lambda \oplus T_\mu = \mathcal{H}$, $\mathfrak{J}T_\lambda = T_\lambda$, $\mathfrak{J}T_\mu = T_\mu$, $JT_\lambda = T_\mu$.

(iii) *If M is a real hypersurface of type C_1 in $SU_{2,m}/S(U_2U_m)$, then $\xi \in \mathfrak{D}^\perp$ at each point of M , and M has three distinct constant principal curvatures*

$$\alpha = 2\sqrt{-2c}, \quad \beta = \sqrt{-2c}, \quad \mu = 0$$

with some $r > 0$. The corresponding principal curvature spaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathcal{H}(-1) \oplus (\mathfrak{D}^\perp \ominus \mathbb{R}\xi), \quad T_\mu = \mathcal{H}(1).$$

(vi) *If M is a real hypersurface of type C_2 in $SU_{2,m}/S(U_2U_m)$, then $\xi \in \mathfrak{D}$ at each point of M and M has three distinct constant principal curvatures*

$$\alpha = 2\sqrt{-c}, \quad \gamma = 0, \quad \lambda = \sqrt{-c}$$

with some $r > 0$. The corresponding principal curvature spaces are

$$T_\alpha = \mathbb{R}\xi \oplus \mathfrak{D}^\perp, \quad T_\gamma = \mathfrak{J}\xi, \quad T_\lambda = \mathcal{H}.$$

A real hypersurface M in a Kähler manifold is said to be *Hopf* if the Reeb vector field ξ is principal. The principal curvature $\alpha = g(A\xi, \xi)$ is called the *Reeb principal curvature* for a Hopf hypersurface M .

Theorem 3.5 ([9, 13]). *Let M be a connected Hopf hypersurface in $\hat{M}^m(c)$, $m \geq 3$. Then $\xi \in \mathfrak{D}$ if and only if*

- (i) *for $c > 0$: M is an open part of a real hypersurface of type B; or*
- (ii) *for $c < 0$: One of the cases (B), (C_2) and (D) in Theorem 3.3 holds.*

We state the following lemma without proof as its proof is entirely similar to that of [8, Theorem 1.5].

Lemma 3.10. *Let M be a connected real hypersurface in $\hat{M}^m(c)$, $m \geq 3$. If $A\mathfrak{D} \subset \mathfrak{D}$ and $\xi \in \mathfrak{D}^\perp$, then M is Hopf.*

By Theorem 3.3 and Lemma 3.10, we have

Theorem 3.6. *Let M be a connected real hypersurface in $\hat{M}^m(c)$, $m \geq 3$. Then $A\mathfrak{D} \subset \mathfrak{D}$ and $\xi \in \mathfrak{D}^\perp$ if and only if*

- (i) *for $c > 0$: M is an open part of a real hypersurface of type A given in Theorem 3.1; or*
- (ii) *for $c < 0$: M is an open part of one of real hypersurfaces of type A or C_1 given in Theorem 3.3.*

4 Hopf hypersurfaces in $\hat{M}^m(c)$

In this section, we shall derive some fundamental properties for Hopf hypersurfaces in $\hat{M}^m(c)$. Suppose M is a Hopf hypersurface in $\hat{M}^m(c)$ with $A\xi = \alpha\xi$. Then as derived in [3, 4, 6], we have

$$d\alpha = (\xi\alpha)\eta - 4c\eta^\perp\phi \quad (4.1)$$

$$\begin{aligned} A\phi A - \frac{\alpha}{2}(\phi A + A\phi) - c(\phi + \phi^\perp) \\ = c \sum_{a=1}^3 \{\xi_a \otimes \eta_a \phi + \phi \xi_a \otimes \eta_a\} - 2c(\xi \otimes \eta^\perp \phi + \phi \xi^\perp \otimes \eta). \end{aligned} \quad (4.2)$$

It follows from (4.1) that

$$d(\xi\alpha) \wedge \eta + (\xi\alpha)d\eta - 4cd(\eta^\perp\phi) = 0. \quad (4.3)$$

This implies that

$$X\xi\alpha = (\xi\xi\alpha)\eta(X) + (d(\xi\alpha) \wedge \eta)(X, \xi) = (\xi\xi\alpha)\eta(X) + 4c(\eta^\perp(AX) - \alpha\eta^\perp(X))$$

or equivalently

$$d(\xi\alpha) = (\xi\xi\alpha)\eta + 4c(\eta^\perp A - \alpha\eta^\perp). \quad (4.4)$$

Combining (4.3)–(4.4), gives

$$4c(\eta^\perp A - \alpha\eta^\perp) \wedge \eta + (\xi\alpha)d\eta - 4cd(\eta^\perp\phi) = 0. \quad (4.5)$$

The following lemma is essentially [2, Lemma 4.2] and [6, Lemma 3.2] but with some additional information.

Lemma 4.1. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$. If $\xi\alpha = 0$, then*

- (a) $A\xi^\perp = \alpha\xi^\perp$, $A\phi^2\xi^\perp = \alpha\phi^2\xi^\perp$,
- (b) $\alpha A\phi\xi^\perp = (\alpha^2 + 4c - 4c\|\xi^\perp\|^2)\phi\xi^\perp$.

Proof. By (4.4), we obtain $\eta^\perp A - \alpha\eta^\perp$; equivalently, $A\xi^\perp = \alpha\xi^\perp$. Next, $A\phi^2\xi^\perp = A(\|\xi^\perp\|^2\xi - \xi^\perp) = \alpha\phi^2\xi^\perp$. Finally, we can obtain (b) by using (a) and (4.2). \square

Now we shall derive some properties of the principal curvatures and their corresponding principal directions for a Hopf hypersurface in $\hat{M}^m(c)$. Observe that we can derive the following two equations from (4.2).

$$\begin{aligned} A\phi A\phi - \frac{\alpha}{2}(\phi A\phi - A) - c(\phi^2 + \theta) &= c \sum_{a=1}^3 \{-\xi_a \otimes \eta_a + \phi\xi_a \otimes \eta_a\phi\} \\ &\quad + c(2\xi \otimes \eta^\perp + 2\xi^\perp \otimes \eta) + \left\{ \frac{\alpha^2}{2} - 2c\|\xi^\perp\|^2 \right\} \xi \otimes \eta \\ \phi A\phi A + \frac{\alpha}{2}(A - \phi A\phi) - c(\phi^2 + \theta) &= c \sum_{a=1}^3 \{\phi\xi_a \otimes \eta_a\phi - \xi_a \otimes \eta_a\} \\ &\quad + c(2\xi \otimes \eta^\perp + 2\xi^\perp \otimes \eta) + \left\{ \frac{\alpha^2}{2} - 2c\|\xi^\perp\|^2 \right\} \xi \otimes \eta. \end{aligned}$$

These two equations imply that

$$A(\phi A\phi) = (\phi A\phi)A. \quad (4.6)$$

Hence there exists a local orthonormal frame $\{X_0 = \xi, X_1, \dots, X_{4m-2}\}$ such that $AX_j = \lambda_j X_j$ and $\phi A\phi X_j = -\mu_j X_j$ for $j \in \{1, \dots, 4m-2\}$. With this setting, (4.2) gives

Lemma 4.2. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$. Then there exists a local orthonormal frame $\{X_0 = \xi, X_1, \dots, X_{4m-2}\}$ such that $AX_j = \lambda_j X_j$ and $A\phi X_j = \mu_j \phi X_j$ for $j \in \{1, \dots, 4m-2\}$. Furthermore, for each $x \in M$ with $\|\xi^\perp\| > 0$, we have*

$$0 = \left\{ \lambda_j \mu_j - \frac{\alpha}{2}(\lambda_j + \mu_j) - c + c\|\xi^\perp\| \right\} X_j^+ \quad (4.7)$$

$$0 = \left\{ \lambda_j \mu_j - \frac{\alpha}{2}(\lambda_j + \mu_j) - c - c\|\xi^\perp\| \right\} X_j^- \quad (4.8)$$

$$0 = \left\{ \lambda_j \mu_j - \frac{\alpha}{2}(\lambda_j + \mu_j) - 2c \right\} g(X_j, \xi_a) + 2c\eta_a(\xi)g(X_j, \xi^\perp) \quad (4.9)$$

$$0 = \left\{ \lambda_j \mu_j - \frac{\alpha}{2}(\lambda_j + \mu_j) - 2c \right\} g(X_j, \phi\xi_a) + 2c\eta_a(\xi)g(X_j, \phi\xi^\perp) \quad (4.10)$$

$$0 = \left\{ \lambda_j \mu_j - \frac{\alpha}{2}(\lambda_j + \mu_j) - 2c + 2c\|\xi^\perp\|^2 \right\} g(X_j, \phi\xi^\perp) \quad (4.11)$$

$$0 = \left\{ \lambda_j \mu_j - \frac{\alpha}{2}(\lambda_j + \mu_j) - 2c + 2c\|\xi^\perp\|^2 \right\} g(X_j, \xi^\perp) \quad (4.12)$$

where X_j^+ and X_j^- is the component of X_j in $\mathcal{H}(1)$ and $\mathcal{H}(-1)$ respectively.

Lemma 4.3. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$. If $0 < \|\xi^\perp\| < 1$, then*

- (a) $A\mathcal{H}(1) \subset \mathcal{H}(1)$,
- (b) $A\mathcal{H}(-1) \subset \mathcal{H}(-1)$,
- (c) $A(\mathbb{R}\phi\xi^\perp \oplus \mathbb{R}\phi^2\xi^\perp) \subset \mathbb{R}\phi\xi^\perp \oplus \mathbb{R}\phi^2\xi^\perp$.

Proof. Consider the local orthonormal frame stated in Lemma 4.2.

- (a) Suppose $X_j^+ \neq 0$ for some $j \in \{1, \dots, 4m-2\}$. Then (4.7)–(4.12) give

$$\begin{aligned} 0 &= -2c\|\xi^\perp\|X_j^- \\ 0 &= -c(\|\xi^\perp\| + 1)g(X_j, \xi_a) + 2c\eta_a(\xi)g(X_j, \xi^\perp) \\ 0 &= -c(\|\xi^\perp\| + 1)g(X_j, \phi\xi_a) + 2c\eta_a(\xi)g(X_j, \phi\xi^\perp) \\ 0 &= c(2\|\xi^\perp\| + 1)(\|\xi^\perp\| - 1) \left\{ g(X_j, \phi\xi^\perp)^2 + g(X_j, \xi^\perp)^2 \right\}. \end{aligned}$$

These imply that $X_j \in \mathcal{H}(1)$ and so we obtain $A\mathcal{H}(1) \subset \mathcal{H}(1)$.

- (b) Suppose $X_j^- \neq 0$ for some $j \in \{1, \dots, 4m-2\}$. If $\|\xi^\perp\| \neq 1/2$ at a point x , then $X_j \perp \phi\xi^\perp, \xi^\perp$ by (4.8), (4.11)–(4.12). Furthermore, since $\|\xi^\perp\| \neq 1$, we obtain $A\mathcal{H}(-1) \subset \mathcal{H}(-1)$ at x by (4.9)–(4.10).

Now suppose $\|\xi^\perp\| = 1/2$ on an open subset $G \subset M$. Then $A\phi\xi^\perp = 0$ by virtue of Lemma 3.4(f). It follows further on from (4.2) that $A\phi^2\xi^\perp = -(4c/\alpha)(1 - \|\xi^\perp\|^2)\phi^2\xi^\perp$. Hence we can select another orthonormal frame in which $X_{4m-3} = (4/\sqrt{3})\phi\xi^\perp, X_{4m-2} = (4/\sqrt{3})\phi^2\xi^\perp$. It follows that for $j \in \{1, \dots, 4m-4\}$, (4.8)–(4.10) imply that if $X_j^- \neq 0$, then $X_j \perp \xi_a, \phi\xi_a$ on G . Hence, we conclude that $A\mathcal{H}(-1) \subset \mathcal{H}(-1)$.

- (c) Since \mathcal{H} is invariant under ϕ and A , we can reconstruct the local orthonormal frame such that X_1, \dots, X_6 (resp. X_7, \dots, X_{4m-1}) are tangent to \mathcal{H}^\perp (resp. \mathcal{H}). Taking the vectors ξ_1, ξ_2, ξ_3 with properties (3.6), (4.9)–(4.12) give

$$\begin{aligned} 0 &= \left\{ \lambda_j\mu_j - \frac{\alpha}{2}(\lambda_j + \mu_j) - 2c \right\} \{g(X_j, \xi_a)^2 + g(X_j, \phi\xi_a)^2\}, \quad a \in \{2, 3\} \\ 0 &= \left\{ \lambda_j\mu_j - \frac{\alpha}{2}(\lambda_j + \mu_j) - 2c + 2c\eta_1(\xi)^2 \right\} \{g(X_j, \phi\xi_1)^2 + g(X_j, \xi_1)^2\}. \end{aligned}$$

These imply that $A(\mathbb{R}\phi\xi^\perp \oplus \mathbb{R}\phi^2\xi^\perp) \perp \xi_a, \phi\xi_a$ for $a \in \{2, 3\}$ and so the desired result is obtained. \square

Lemma 4.4. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$ such that $0 < \|\xi^\perp\| < 1$ on M . Suppose $X \in \mathcal{H}(\varepsilon)$ such that $AX = \lambda X$ and $A\phi X = \mu\phi X$. Then*

$$\nabla_X \phi\xi^\perp = \lambda\|\xi^\perp\|(\varepsilon + \|\xi^\perp\|)X; \quad \nabla_X \phi^2\xi^\perp = \lambda\|\xi^\perp\|(\varepsilon + \|\xi^\perp\|)\phi X.$$

Furthermore, if we put

$$A\phi\xi^\perp = u\phi\xi^\perp - v\phi^2\xi^\perp; \quad A(-\phi^2\xi^\perp) = p\phi\xi^\perp - q\phi^2\xi^\perp, \quad (4.13)$$

then for any $Y \in \mathcal{H}$

$$\begin{aligned} g((\nabla_X A)\phi\xi^\perp, Y) &= \lambda\|\xi^\perp\|(\varepsilon + \|\xi^\perp\|)\{(u - \lambda)g(X, Y) - vg(\phi X, Y)\} \\ g((\nabla_X A)\phi^2\xi^\perp, Y) &= \lambda\|\xi^\perp\|(\varepsilon + \|\xi^\perp\|)\{-pg(X, Y) + (q - \mu)g(\phi X, Y)\}. \end{aligned}$$

Proof. Note that $\theta X = \varepsilon\|\xi^\perp\|X$ and $\phi^\perp X = -\varepsilon\|\xi^\perp\|\phi X$. Then by Lemma 3.4 we obtain

$$\nabla_X \phi^2\xi^\perp = \nabla_X(-\xi^\perp + \|\xi^\perp\|^2\xi) = -\phi^\perp AX + \|\xi^\perp\|^2\phi AX = \lambda\|\xi^\perp\|(\varepsilon + \|\xi^\perp\|)\phi X.$$

The other can be obtained similarly. Next under the setting of (4.13), we obtain

$$\begin{aligned} g((\nabla_X A)\phi\xi^\perp, Y) &= g(u\nabla_X \phi\xi^\perp - v\nabla_X \phi^2\xi^\perp - A\nabla_X \phi\xi^\perp, Y) \\ &= \lambda\|\xi^\perp\|(\varepsilon + \|\xi^\perp\|)\{(u - \lambda)g(X, Y) - vg(\phi X, Y)\}. \end{aligned}$$

The last identity can be obtained by a similar calculation. \square

5 Mixed foliate real hypersurfaces in $\hat{M}^m(c)$

Let M be a submanifold in a Kähler manifold \hat{M} . If the dimension of the maximal holomorphic subspace $\mathcal{C}_x = JT_x M \cap T_x M$, $x \in M$ is constant and its orthogonal complementary distribution \mathcal{C}^\perp in TM is totally real, then M is called a *CR-submanifold*. If $\dim \mathcal{C} \neq 0$ and $\dim \mathcal{C}^\perp \neq 0$, then the CR-submanifold M is said to be *proper*. A CR-submanifold M is said to be *mixed totally geodesic* if $h(\mathcal{C}, \mathcal{C}^\perp) = 0$, where h is the second fundamental form of M . A *mixed foliate* CR-submanifold M is a mixed totally geodesic CR-submanifold such that the distribution \mathcal{C} is integrable (cf. [5]).

A real hypersurface is a typical example of a proper CR-submanifold in a Kähler manifold with $\mathcal{C}^\perp = \mathbb{R}\xi$. It is clear that M is mixed totally geodesic if and only if it is Hopf. Furthermore by a result in [5], we can state

Lemma 5.1. *Let M be a real hypersurface in a Kähler manifold. Then M is mixed foliate if and only if $\phi A + A\phi = 0$.*

In this section, we shall prove the nonexistence of mixed foliate real hypersurfaces in $\hat{M}^m(c)$.

Theorem 5.1. *There does not exist any mixed foliate real hypersurface in $\hat{M}^m(c)$, $m \geq 3$.*

Remark 5.1. *The nonexistence of mixed foliate real hypersurfaces in a non-flat complex space form was obtained in [7].*

The proof of Theorem 5.1 is splitted into several parts. We first prove

Lemma 5.2. *Let M be a real hypersurface in $\hat{M}^m(c)$. Then $\phi A + A\phi \neq 0$ on each open subset $G \subset M$ with $\xi^\perp = 0$.*

Proof. Since $\xi^\perp = 0$ on G , $A\phi\xi_a = 0$ for $a \in \{1, 2, 3\}$ by Lemma 3.1. Suppose $\phi A + A\phi = 0$ on G . Then $A\xi_a = \phi A\phi\xi_a = 0$ and $A\xi = \alpha\xi$; so G is an open part of one of the real hypersurfaces given in Theorem 3.1 and Theorem 3.3. However, along the direction ξ_a , the principal curvature is non-zero for these real hypersurfaces according to Theorem 3.2 and Theorem 3.4; a contradiction and so the result is obtained. \square

We observe that if $\phi A + A\phi = 0$, then M is Hopf. Moreover, for each $x \in M$ with $\|\xi^\perp\| > 0$, by taking a principal curvature vector X_j in Lemma 4.2 with $X_j^- \neq 0$, we obtain $-\lambda_j^2 - c(1 + \|\xi^\perp\|) = 0$ from (4.8) and so $c < 0$. It follows that we only need to consider the case $c < 0$. We shall prepare some results before proceeding to the proof of Theorem 5.1.

Lemma 5.3. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$. Suppose $G \subset M$ is an open set with $0 < \|\xi^\perp\| < 1$. If $(\phi A + A\phi)\mathcal{H}(-1) = 0$ on G , then for $Y, Z, W \in \mathcal{H}(-1)$ on G , we have*

- (a) $A^2Y = -c(1 + \|\xi\|^\perp)Y$,
- (b) $g((\nabla_Y A)Z, W) = 0$.

Proof. (a) It can be obtained directly from (4.2) and the fact $\phi^\perp Y = \|\xi^\perp\|\phi Y$ for $Y \in \mathcal{H}(-1)$.

(b) For all vector fields Y, Z, W tangent to $\mathcal{H}(-1)$ on G , it follows from (a), Lemma 3.4(f) and the Codazzi equation that

$$\begin{aligned} 0 &= g((\nabla_W A)Y, AZ) + g((\nabla_W A)Z, AY) + g(A^2\nabla_W Y, Z) + g(A^2Y, \nabla_W Z) \\ &\quad + c(1 + \|\xi^\perp\|)\{g(\nabla_W Y, Z) + g(Y, \nabla_W Z)\} \\ &= g((\nabla_W A)Y, AZ) + g((\nabla_Z A)W, AY) \end{aligned}$$

By taking a cyclic sum over Y, Z, W in the preceding equation, and then subtracting the obtained equation from the preceding equation, yields

$$g((\nabla_Y A)Z, AW) = 0, \quad Y, Z, W \in \mathcal{H}(-1).$$

By (a) and Lemma 4.3(b), A is an isomorphism when restricted to $\mathcal{H}(-1)$. Hence, we obtain the lemma. \square

Lemma 5.4. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$. Suppose $G \subset M$ is an open set with $\|\xi^\perp\| = 1$. If $(\phi A + A\phi)\mathcal{W} = 0$ on G , where $\mathcal{W} = \mathcal{H}(-1) \oplus (\mathfrak{D}^\perp \ominus \mathbb{R}\xi)$, then for $Y, Z, W \in \mathcal{W}$ on G , we have*

- (a) $A^2Y = -2cY$,
- (b) $g((\nabla_Y A)Z, W) = 0$.

Proof. (a) It can be obtained directly from (4.2).

(b) Using a similar manner as in the proof of Lemma 5.3 but with the help of Lemma 3.7, we obtain

$$g((\nabla_Y A)Z, AW) = 0, \quad Y, Z, W \in \mathcal{W}.$$

We note that $AW \subset \mathcal{W}$ by virtue of Lemma 3.6, together with (a), we obtain that $A|_{\mathcal{W}}$ is an isomorphism. Hence, we obtain the lemma. \square

Lemma 5.5. *Let M be a Hopf hypersurface M in $\hat{M}^m(c)$ and $\|\xi^\perp\| > 0$ at a point $x \in M$. Suppose \mathcal{V} is a subspace of $\mathcal{H}(-1)$ at x that is invariant under A and ϕ . If $(\phi A + A\phi)\mathcal{V} = 0$, then*

$$\sum_{j=1}^n g((R(e_j, \phi e_j)A)Z, W) = -4c(5 + \|\xi^\perp\| + 2n)g(\phi AZ, W)$$

for any $Z, W \in \mathcal{V}$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of \mathcal{V} and $n = \dim \mathcal{V}$.

Proof. We take ξ_1, ξ_2, ξ_3 with properties (3.6). Then under this situation, $\phi_1 X = \phi X$ and $\phi_b X, \theta_b X \in \mathcal{H}_1(1)$, $b \in \{2, 3\}$, for $X \in \mathcal{H}(-1)$. It follows from the Gauss equation and Lemma 5.3–5.4 that

$$\begin{aligned} g((R(X, Y)A)Z, W) = & c(3 + \|\xi^\perp\|)\{g(Y, AZ)g(X, W) - g(X, AZ)g(Y, W) \\ & - g(Y, Z)g(AX, W) + g(X, Z)g(AY, W)\} \\ & + 2c\{g(\phi Y, AZ)g(\phi X, W) - g(\phi X, AZ)g(\phi Y, W) \\ & - g(\phi Y, Z)g(A\phi X, W) + g(\phi X, Z)g(A\phi Y, W) \\ & - 4g(\phi X, Y)g(\phi AZ, W)\} \end{aligned}$$

for any $X, Y, Z, W \in \mathcal{V}$. Hence the lemma can be obtained directly from the preceding equation. \square

Lemma 5.6. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$. Suppose $G \subset M$ is an open set with $0 < \|\xi^\perp\| < 1$. If $(\phi A + A\phi)\mathcal{H} = 0$ on G , then $\nabla_X Z \perp \mathcal{H}(1)$ for all vector fields X, Z tangent to $\mathcal{H}(-1)$ on G .*

Proof. By the hypothesis $(\phi A + A\phi)\mathcal{H} = 0$ and Lemma 4.2–4.3, we can select local orthonormal principal vector fields $X_1, X_2, \dots, X_{4m-8}$ such that $X_j, X_{m-2+j} = \phi X_j$ are tangent to $\mathcal{H}(-1)$ with $\lambda_j = \lambda = -\lambda_{m-2+j}$, where $\lambda = \sqrt{-c(1 + \|\xi^\perp\|)}$, and $X_{2m-4+j}, \dots, X_{3m-6+j} = \phi X_{2m-4+j}$ are tangent to $\mathcal{H}(1)$, $j \in \{1, \dots, m-2\}$. We can further deduce from (4.7)–(4.8) that $\lambda_r \neq \lambda$ for $r \in \{2m-3, \dots, 4m-8\}$.

Fixed $i, j \in \{1, \dots, m-2\}$ and $r \in \{2m-3, \dots, 4m-8\}$, It follows from the Codazzi equation and Lemma 3.4(f) that

$$g(\nabla_{X_i} X_j, X_r) = -\frac{1}{(\lambda_r - \lambda)} g(\nabla_{X_i} A)X_r - (\nabla_{X_r} A)X_i, X_j = 0.$$

We can further deduce from the preceding equation that

$$g(\nabla_{X_i} \phi X_j, X_r) = g((\nabla_{X_i} \phi)X_j, X_r) - g(\nabla_{X_i} X_j, \phi X_r) = 0.$$

Similarly, we have $g(\nabla_{\phi X_i} \phi X_j, X_r) = g(\nabla_{\phi X_i} X_j, X_r) = 0$. This completes the proof. \square

For a real hypersurface M in $\hat{M}^m(c)$, if $G \subset M$ is an open set with $\|\xi^\perp\| = 1$, then $A\mathcal{H}(1) = 0$ (and so $(\phi A + A\phi)\mathcal{H}(1) = 0$) on G according to Lemma 3.6. Based on this observation, although there is a slight difference between the hypotheses, the following lemma can be obtained in a similar manner as in the proof of Lemma 5.6.

Lemma 5.7. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$. Suppose $G \subset M$ is an open set with $\|\xi^\perp\| = 1$ and $\mathcal{V} \subset \mathcal{H}(-1)$ is a subbundle over G that is invariant under A and ϕ . If $(\phi A + A\phi)\mathcal{V} = 0$ on G , then $\nabla_X Z \perp \mathcal{H}(1)$ for all vector fields X, Z tangent to \mathcal{V} on G .*

Lemma 5.8. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$, $m \geq 3$. Then $(\phi A + A\phi)\mathcal{H} \neq 0$ on each open subset $G \subset M$ with $0 < \|\xi^\perp\| < 1$.*

Proof. Suppose $(\phi A + A\phi)\mathcal{H} = 0$ on G . Then by Lemma 5.3, we obtain

$$\begin{aligned} &g((R(X, Y)A)Z, W) + g((\nabla_Y A)\nabla_X Z, W) + g((\nabla_Y A)Z, \nabla_X W) \\ &+ g((\nabla_{[X, Y]}A)Z, W) - g((\nabla_X A)\nabla_Y Z, W) - g((\nabla_X A)Z, \nabla_Y W) = 0 \end{aligned} \quad (5.1)$$

for any vector fields X, Y, Z, W tangent to $\mathcal{H}(-1)$, here we have used the fact

$$(R(X, Y)A)Z = \nabla^2 A(; Y; X)Z - \nabla^2 A(; X; Y)Z$$

where

$$\nabla^2 A(; Y; X)Z := \nabla_X(\nabla_Y A)Z - (\nabla_{\nabla_X Y} A)Z - (\nabla_Y A)\nabla_X Z.$$

By using Lemma 3.4(h)–(j), Lemma 3.8, Lemma 5.3 and Lemma 5.6, on one hand, we obtain

$$g((\nabla_{[X, Y]}A)Z, W) = 0 \quad (5.2)$$

and on the other hand

$$\begin{aligned} &g((\nabla_Y A)\nabla_X Z, W) \\ &= \eta(\nabla_X Z)g((\nabla_Y A)\xi, W) + \frac{g(\nabla_X Z, \phi^2 \xi^\perp)}{\|\xi^\perp\|^2(1 - \|\xi^\perp\|^2)}g((\nabla_Y A)\phi^2 \xi^\perp, W) \\ &\quad + \frac{g(\nabla_X Z, \phi \xi^\perp)}{\|\xi^\perp\|^2(1 - \|\xi^\perp\|^2)}g((\nabla_Y A)\phi \xi^\perp, W) \\ &= -g(\phi AX, Z)g((\nabla_Y A)\xi, W) + \frac{g(\phi AX, Z)}{\|\xi^\perp\|(1 + \|\xi^\perp\|)}g((\nabla_Y A)\phi^2 \xi^\perp, W) \\ &\quad + \frac{g(AX, Z)}{\|\xi^\perp\|(1 + \|\xi^\perp\|)}g((\nabla_Y A)\phi \xi^\perp, W) \end{aligned} \quad (5.3)$$

for any vector fields X, Y, Z, W tangent to $\mathcal{H}(-1)$.

Let $\{e_1, \dots, e_{2m-4}\}$ be an orthonormal basis of $\mathcal{H}(-1)$ and Z be a unit vector field tangent to $\mathcal{H}(-1)$ such that $AZ = \lambda Z$ (and so $A\phi Z = -\lambda\phi Z$), where $\lambda =$

$\sqrt{-c(1 + \|\xi^\perp\|)}$. Then by using (5.1)–(5.3) and Lemma 4.4, we obtain

$$\begin{aligned}
& \sum_{j=1}^{2m-4} ((R(e_j, \phi e_j)A)Z, \phi Z) \\
&= -2\lambda g((\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z, \xi) + \frac{2\lambda g((\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z, \phi^2 \xi^\perp)}{\|\xi^\perp\|(1 + \|\xi^\perp\|)} \\
&\quad - 2\lambda \frac{g((\nabla_{\phi Z} A)\phi \xi^\perp, \phi Z) + g(\nabla_Z A)\phi \xi^\perp, Z)}{\|\xi^\perp\|(1 + \|\xi^\perp\|)} \\
&= -2\lambda \{-2c(1 + \|\xi^\perp\|)\} + 2\lambda \{2c(1 - \|\xi^\perp\|)\} - 2\lambda \{-2c(1 - \|\xi^\perp\|)\} \\
&= 4c\lambda(3 - \|\xi^\perp\|).
\end{aligned}$$

This, together with Lemma 5.5, gives $16c\lambda m = 0$. This is a contradiction and so the proof is completed. \square

Lemma 5.9. *Let M be a real hypersurface in $\hat{M}^m(c)$, $m \geq 3$. Then $\phi A + A\phi \neq 0$ on each open subset $G \subset M$.*

Proof. Suppose $\phi A + A\phi = 0$ on an open subset $G \subset M$. By virtue of Lemma 5.2 and Lemma 5.8, $\|\xi^\perp\| = 1$ on G or $\xi = \xi^\perp \in \mathfrak{D}^\perp$. We consider the vectors ξ_1, ξ_2, ξ_3 with properties (3.6)–(3.7).

We first prove that $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$. It suffices to show that $A\xi_2, A\xi_3 \in \mathfrak{D}^\perp$. Suppose $A\xi_2 = p\xi_2 + q\xi_3 + rU$, where U is a unit vector field tangent to $\mathcal{H}(-1)$ and r is a non-vanishing function on G . Then by the hypothesis $\phi A + A\phi = 0$ and Lemma 5.4, we can obtain

$$\begin{aligned}
A\xi_3 &= q\xi_2 - p\xi_3 + r\phi U \\
AU &= r\xi_2 - pU - q\phi U \\
A\phi U &= r\xi_3 - qU + p\phi U.
\end{aligned}$$

It follows that $\mathcal{V} := \mathcal{H}(-1) \ominus \text{Span}\{U, \phi U\}$ is invariant under A and ϕ . By Lemma 5.4, we obtain

$$\begin{aligned}
& g((R(X, Y)A)Z, W) + g((\nabla_Y A)\nabla_X Z, W) + g((\nabla_Y A)Z, \nabla_X W) \\
&+ g((\nabla_{[X, Y]}A)Z, W) - g((\nabla_X A)\nabla_Y Z, W) - g((\nabla_X A)Z, \nabla_Y W) = 0
\end{aligned}$$

for any vector fields X, Y, Z, W tangent to \mathcal{V} . On the other hand, by Lemma 5.4 and Lemma 5.7, we obtain

$$\begin{aligned}
g((\nabla_Y A)\nabla_X Z, W) &= \eta(\nabla_X Z)g((\nabla_Y A)\xi, W) \\
&= -g(\phi AX, Z)g(\alpha\phi AY - 2c\phi Y, W)
\end{aligned}$$

for any vector fields X, Y, Z, W tangent to \mathcal{V} . By using these two equations, we obtain

$$\sum_{j=1}^{2m-4} ((R(e_j, \phi e_j)A)Z, W) - 8cg(\phi AZ, W) = 0, \quad Z, W \in \mathcal{V}$$

where $\{e_1, \dots, e_{2m-4}\}$ is an orthonormal basis for \mathcal{V} . This, together with Lemma 5.5, gives

$$-16cmg(\phi AZ, W) = 0, \quad Z, W \in \mathcal{V}.$$

This contradicts the fact that A is an isomorphism on \mathcal{V} . Hence $A\xi_2 \in \mathfrak{D}^\perp$ and so $A\xi_3 = \phi A\xi_2 \in \mathfrak{D}^\perp$. Accordingly, $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$. It follows that G is an open part of one of the real hypersurfaces stated in Theorem 3.1 and Theorem 3.3. However, the fact $(\phi A + A\phi)\xi_a = 0$ prevents M from being any one of the cases in these two theorems in light of Theorem 3.2 and Theorem 3.4; it is a contradiction and so the proof is completed. \square

Proof of Theorem 5.1. It is an immediate consequence of Lemma 5.9. \square

By using Theorem 5.1, we obtain the following general properties of Hopf hypersurfaces.

Theorem 5.2. *Let M be a Hopf hypersurface in $\hat{M}^m(c)$, $m \geq 3$. Then*

- (a) $\xi\alpha = 0$; $d\alpha = -4c\eta^\perp\phi$,
- (b) α is constant if and only if either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$,
- (c) $A\xi^\perp = \alpha\xi^\perp$, $A\phi^2\xi^\perp = \alpha\phi^2\xi^\perp$,
- (d) $\alpha A\phi\xi^\perp = (\alpha^2 + 4c - 4c\|\xi^\perp\|^2)\phi\xi^\perp$.

Proof. (a) In each open subset $G \subset M$ with $\|\xi^\perp\| = 0$, $\theta = \phi^\perp = 0$ and so $\xi \in \mathfrak{D}$. Then one of the cases in Theorem 3.5 occur, and so α is constant on G ; this gives $\xi\alpha = 0$ on G . Next, for each $x \in M$ with $0 < \|\xi^\perp\| < 1$, it follows from (3.8), (4.5) and Lemma 4.3 that

$$(\xi\alpha)g((\phi A + A\phi)X, Y) = (\xi\alpha)d\eta(X, Y) = 0, \quad X, Y \in \mathcal{H}.$$

Hence, we obtain $\xi\alpha = 0$ at x by Lemma 5.8. Now consider an open subset $G \subset M$ with $\|\xi^\perp\| = 1$. Then $\eta = \eta^\perp$ and so (4.5) descends to $(\xi\alpha)d\eta = 0$. It follows from Lemma 5.9 that $\xi\alpha = 0$ on G . Consequently, we conclude that $\xi\alpha = 0$ on M . Next by (4.1), we obtain $d\alpha = -4c\eta^\perp\phi$.

(b)–(d) These can be obtained immediately from (a) and Lemma 4.1. \square

With the help of Theorem 5.2, we can complete the classification problem of contact real hypersurfaces in $\hat{M}^m(c)$, $c < 0$, considered by Berndt, Lee and Suh in [2].

Recall that a real hypersurface M in a Kähler manifold is said to be *contact* if $\phi A + A\phi = \rho\phi$ for a nowhere zero function ρ on M . This means that the almost contact metric structure (ϕ, ξ, η, g) of M is contact up to a \mathcal{C} -homothetic deformation. By using Theorem 5.2 and [2, Theorem 1.1], we obtain the following result.

Theorem 5.3. *Let M be a real hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then M is contact if and only if it is an open part of one of real hypersurfaces of type B or C_2 given in Theorem 3.3.*

Remark 5.2. *Theorem 5.3 was obtained in [13] for the case $c > 0$.*

6 q -umbilical real hypersurfaces in $\hat{M}^m(c)$

Recall that a real hypersurfaces M in $\hat{M}^m(c)$ is said to be q -umbilical if it satisfies

$$A = f_1 \mathbb{I} + f_2 \theta + f_3 \sum_{a=1}^3 \xi_a \otimes \eta_a$$

where f_1, f_2, f_3 are functions on M . This class of real hypersurfaces is interesting to be studied as it includes three important types of real hypersurfaces. We can easily obtain the following from Theorem 3.2 and Theorem 3.4.

Lemma 6.1.

(i) Real hypersurfaces of type A in $SU_{m+2}/S(U_2U_m)$ are q -umbilical with

$$f_1 = -f_2 = -\frac{\sqrt{2c} \tan(\sqrt{2cr})}{2}, \quad f_3 = \sqrt{2c} \cot(\sqrt{2cr}), \quad 0 < r < \frac{\pi}{\sqrt{8c}}, \quad c > 0.$$

(ii) Real hypersurfaces of type A in $SU_{2,m}/S(U_2U_m)$ are q -umbilical with

$$f_1 = -f_2 = \frac{\sqrt{-2c} \tanh(\sqrt{-2cr})}{2}, \quad f_3 = \sqrt{-2c} \coth(\sqrt{-2cr}), \quad r > 0, \quad c < 0.$$

(iii) Real hypersurfaces of type C_1 in $SU_{2,m}/S(U_2U_m)$ are q -umbilical with

$$f_1 = -f_2 = \frac{\sqrt{-2c}}{2}, \quad f_3 = \sqrt{-2c}, \quad c < 0.$$

We shall consider a more general condition than q -umbilicity to classify q -umbilical real hypersurfaces as well as to obtain a nonexistence result.

Theorem 6.1. Let M be a connected real hypersurface in $\hat{M}^m(c)$, $m \geq 3$. Suppose M satisfies

$$A = f_1 \mathbb{I} + f_2 \theta + f_3 \sum_{a=1}^3 \xi_a \otimes \eta_a + f_4 \xi \otimes \eta \quad (6.1)$$

where f_1, f_2, f_3, f_4 are functions on M . Then $f_4 = 0$, that is, M is q -umbilical. Furthermore, one of the following holds:

- (i) for $c > 0$: M is an open part of a real hypersurface of type A given in Theorem 3.1;
or
- (ii) for $c < 0$: M is an open part of one of real hypersurfaces of type A or C_1 given in Theorem 3.3.

Proof. For each $x \in M$ with $\|\xi^\perp\| > 0$, by (6.1), we obtain

$$\left. \begin{aligned} AX &= (f_1 + \varepsilon f_2 \|\xi^\perp\|)X, \quad X \in \mathcal{H}(\varepsilon), \quad \varepsilon \in \{1, 2\} \\ A\xi &= (f_1 + f_4)\xi + (f_3 - f_2)\xi^\perp \\ A\xi^\perp &= (f_4 - f_2)\|\xi^\perp\|^2\xi + (f_1 + f_3)\xi^\perp \\ A\phi\xi^\perp &= (f_1 + f_2\|\xi^\perp\|^2)\phi\xi^\perp. \end{aligned} \right\} \quad (6.2)$$

Consider an open subset $G \subset M$ with $0 < \|\xi^\perp\| < 1$. Write $\lambda_\varepsilon = f_1 + \varepsilon f_2 \|\xi^\perp\|$ and let $X, Y \in \mathcal{H}(\varepsilon)$. Then we have

$$\nabla_X \xi = \lambda_\varepsilon \phi X, \quad \nabla_X \xi^\perp = \phi^\perp A X = -\varepsilon \|\xi^\perp\| \lambda_\varepsilon \phi X, \quad (6.3)$$

where we have used the fact $\phi^\perp X = -\varepsilon \|\xi^\perp\| \phi X$. Hence, by (6.2)–(6.3), we obtain

$$g((\nabla_X A)\xi, Y) = \lambda_\varepsilon (f_4 - \varepsilon f_3 \|\xi^\perp\|) g(\phi X, Y)$$

It follows from the preceding equation and the Codazzi equation that

$$\begin{aligned} 2\lambda_\varepsilon (f_4 - \varepsilon f_3 \|\xi^\perp\|) g(\phi X, Y) &= g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ &= -2c(1 - \varepsilon \|\xi^\perp\|) g(\phi X, Y), \end{aligned}$$

which gives

$$f_1 f_4 - f_2 f_3 \|\xi^\perp\|^2 + c = \varepsilon \|\xi^\perp\| \{f_1 f_3 - f_2 f_4 + c\}, \quad \varepsilon \in \{1, -1\}$$

and so

$$f_2 f_3 \|\xi^\perp\|^2 - f_1 f_4 = c = f_2 f_4 - f_1 f_3. \quad (6.4)$$

Similarly, we compute

$$\begin{aligned} 2\lambda_\varepsilon \|\xi^\perp\| (f_4 \|\xi^\perp\| - \varepsilon f_3) g(\phi X, Y) &= g((\nabla_X A)\xi^\perp, Y) - g((\nabla_Y A)\xi^\perp, X) \\ &= -2c \|\xi^\perp\| (\|\xi^\perp\| - \varepsilon) g(\phi X, Y) \end{aligned}$$

to obtain

$$f_2 f_4 \|\xi^\perp\|^2 - f_1 f_3 = c = f_2 f_3 - f_1 f_4. \quad (6.5)$$

We can deduce from (6.4)–(6.5) that

$$f_2 = 0, \quad f_3 = f_4, \quad f_1 f_3 = -c.$$

It follows that $f_1 + f_2 \|\xi^\perp\|^2 = f_1 \neq 0$. For $b \in \{1, 2, 3\}$, by using Lemma 3.5, (3.8) and (6.2), we obtain

$$\begin{aligned} 0 &= (\theta A - A\theta)\xi_b - 2 \sum_{a=1}^3 \{g(A\xi_b, \phi\xi_a)\phi\xi_a - g(\phi\xi_a, \xi_b)A\phi\xi_a\} \\ &= f_3 \left\{ (\eta_{b+2}(\xi)^2 + \eta_{b+1}(\xi)^2)\xi_b - \eta_b(\xi)\eta_{b+1}(\xi)\xi_{b+1} - \eta_b(\xi)\eta_{b+2}(\xi)\xi_{b+2} \right. \\ &\quad \left. - \eta_{b+2}(\xi)\phi\xi_{b+1} + \eta_{b+1}(\xi)\phi\xi_{b+2} \right\}. \end{aligned}$$

Since $f_3 \neq 0$ and $\{\xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3\}$ is linearly independent, we have $\eta_b(\xi) = 0$ for $b \in \{1, 2, 3\}$. Hence G must be empty and so either $\|\xi^\perp\| = 0$ or $\|\xi^\perp\| = 1$ everywhere.

We first consider $\xi^\perp = 0$. Then $\theta = 0$ and $\eta_a(\xi) = 0$, $a \in \{1, 2, 3\}$ in this case. Hence by Lemma 3.1, we obtain $A\phi\xi_a = 0$ and so $f_1 = 0$ by virtue of (6.1). We can

then obtain $A\mathcal{H} = 0$, $A\xi = f_4\xi$ and $A\xi_a = f_3\xi_a$ for $a \in \{1, 2, 3\}$ further on. However, by Theorem 3.1–3.4, we see that such a real hypersurface does not exist. Consequently, this case cannot occur.

Finally, suppose that $\|\xi^\perp\| = 1$, which means that $\xi = \xi^\perp \in \mathfrak{D}^\perp$. This, together with (6.1), gives $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$. Then by Theorem 3.6, M is an open part of one of the spaces listed in the theorem. Furthermore it follows from Lemma 6.1 that $f_4 = 0$. \square

Recall that a real hypersurface M in a Kähler manifold is said to be η -umbilical if it satisfies

$$A = u\mathbb{I} + v\xi \otimes \eta$$

for some functions u, v on M . By Theorem 6.1, we immediately obtain

Corollary 6.1. *There does not exist any η -umbilical real hypersurface M in $\hat{M}^m(c)$, $m \geq 3$.*

References

- [1] J. Berndt, *Riemannian geometry of complex two-plane Grassmannians*, Rend. Semin. Mat. Univ. Politec. Torino **55**(1997), 19–83.
- [2] J. Berndt, H. Lee and Y.J. Suh, *Contact hypersurfaces in noncompact complex Grassmannians of rank two*, Int. J. Math. **24**(2013) 1350089, 11 pp.
- [3] J. Berndt and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians*, Monatsh. Math. **127**(1999), 1–14.
- [4] J. Berndt and Y.J. Suh, *Hypersurfaces in noncompact complex Grassmannians of rank two*, Int. J. Math. **23**(2012) 1250103, 35 pp.
- [5] B.Y. Chen, *CR-submanifolds of a Kaehler manifold II*, J. Diff. Geom. **16**(1981), 493–509.
- [6] I. Jeong, C. Machado, J.D. Pérez and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with \mathfrak{D}^\perp -parallel structure Jacobi operator*, Int. J. Math. **22**(2011), 655–673.
- [7] U.H. Ki and Y.J. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama Univ. **32**(1990), 207–221.
- [8] R.H. Lee and T.H. Loo, *Real hypersurfaces of type A in complex two-plane Grassmannians related to the Reeb vector field*, AIP Conf. Proc. **1682**(2015) 040005, 8 pp.
- [9] H. Lee and Y.J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector*, Bull. Korean Math. Soc. **47**(2010), 551–561.
- [10] T.H. Loo, *Semi-parallel real hypersurfaces in complex two-plane Grassmannians*, Differ. Geom. Appl. **34**(2014), 87–102.
- [11] Y.J. Suh, *Pseudo-Einstein real hypersurfaces in complex two-plane Grassmannians*, Bull. Aust. Math. Soc. **73**(2006), 183–200.

- [12] Y.J. Suh, *Real hypersurfaces in complex hyperbolic two-plane Grassmannians with Reeb vector field*, Adv. Appl. Math. **55**(2014), 131–145.
- [13] Y.J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians*, Monatsh. Math. **147**(2006), 337–355.